## Lesson 19: End Behavior of Rational Functions

* Let’s explore the end behavior of rational functions.

### 19.1: Different Divisions, Revisited

Complete all three representations of the polynomial division following the forms of the integer division.



$\begin{matrix}2x^{2}\\x+12x^{3}+7x^{2}+7x+5\end{matrix}$

$2775=11\left(252\right)+3$

$2x^{3}+7x^{2}+7x+5=$

$\frac{2775}{11}=252+\frac{3}{11}$

$\frac{2x^{3}+7x^{2}+7x+5}{x+1}=$

### 19.2: Combined Fuel Economy

In 2000, the Environmental Protection Agency (EPA) reported a combined fuel efficiency for cars that assumes 55% city driving and 45% highway driving. The expression for the combined fuel efficiency of a car that gets $x$ mpg in the city and $h$ mpg on the highway can be written as $\frac{100xh}{55x+45h}$.

1. Several conventional cars have a fuel economy for highway driving that is about 10 mpg higher than for city driving. That is, $h=x+10$. Write a function $f$ that represents the combined fuel efficiency for cars like these in terms of $x$.
2. Rewrite $f$ in the form $q\left(x\right)+\frac{r\left(x\right)}{b\left(x\right)}$ where $q\left(x\right)$, $r\left(x\right)$, and $b\left(x\right)$ are polynomials.

### 19.3: Exploring End Behavior

| function | degreeof num. | degreeof den. | rewritten in the form of$q\left(x\right)+\frac{r\left(x\right)}{b\left(x\right)}$ | end behavior |
| --- | --- | --- | --- | --- |
| $g\left(x\right)=-\frac{5}{x+2}$ |  |  |  |  |
| $h\left(x\right)=\frac{7x−5}{x+2}$ |  |  |  |  |
| $j\left(x\right)=\frac{3x^{2}+7x−5}{x+2}$ |  |  |  |  |
| $k\left(x\right)=\frac{2x^{3}+3x^{2}+7x−5}{x+2}$ |  |  |  |  |
| $m\left(x\right)=\frac{x+2}{2x^{3}+3x^{2}+7x−5}$ |  |  |  |  |

1. Complete the table to explore the end behavior for rational functions.
2. What do you notice about the end behavior of different types of rational functions?

#### Are you ready for more?

1. Graph $y=j\left(x\right)$ and the line it approaches.
2. Under what conditions would the end behavior of the graph of a rational function approach a line that is not horizontal?
3. Create a rational function that approaches the line $y=2x−3$ as $x$ gets larger and larger in either the positive or negative direction.

### Lesson 19 Summary

In earlier lessons, we saw rational functions whose end behavior could be described by a horizontal asymptote. For example, we can rewrite functions like $d\left(x\right)=\frac{x+4}{x}$ as $d\left(x\right)=1+\frac{4}{x}$ to see more clearly that as $x$ gets larger and larger in either the positive or negative direction, the value of $\frac{4}{x}$ gets closer and closer to 0, which means the value of $d\left(x\right)$ gets closer to 1. We can use similar thinking to understand rational functions that do not have horizontal asymptotes.

For example, consider $f\left(x\right)=\frac{x^{2}+4x+5}{x−3}$. Using division, the expression can be rewritten as $f\left(x\right)=x+7+\frac{26}{x−3}$. As $x$ gets larger and larger in either the positive or negative direction, the value of the term $\frac{26}{x−3}$ gets closer and closer to 0, which means the value of $d\left(x\right)$ gets closer to the value of $x+7$. This means that the end behavior of $f$ can be described by the line $y=x+7$. Here is a graph of $y=f\left(x\right)$, the line $y=x+7$, and the vertical asymptote of the function at $x=3$:





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